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Limit Theorems on Fuzzy Markov Chains

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Abstract: In this paper we attempt to show the limit theorems for fuzzy Markov chains. Using stationary distribution we establish conditions for the existence of a Fuzzy Markov chain.

Key Words*: Fuzzy Markov chain, Fuzzy Transition Probability and Fuzzy functions.*

I. INTRODUCTION

Markov chains are one of the most important tools to model random phenomena evolving in time. A weak point of the most widely used model is that transition probabilities have to be constant and precisely known. An attempt to relax this restriction was proposed By Skulj[8] where the assumption of precisely known initial and transition probabilities is relaxed so that probability intervals are used instead of precise probabilities. Their model is based on the assumption that constant classical probabilities rule the process but only approximations are known instead of precise values.

The theory of Markov systems provide an effective and powerful tool for describing State of the system. Since numerous applied probability models can be adopted in their framework. Roughly speaking the Markov property requires that knowledge of the current state of the system provides all the information relevant to predicting its future. There have been a few other papers published on fuzzy Markov Chains[2,3,5,6].

The organization of the paper is as detailed below Section 2 is devoted to fuzzy functions where Continuous we have defined the fuzzy functions. Section 3 is addressing the notions of limit theorems on Fuzzy Markov chains. In Section 4 we are discussing about stationary distribution of a fuzzy Markov chain. We establish the conditions for the existence of a Markov chain.

II. FUZZY FUNCTIONS

Set valued functions and their calculus were found useful in of the problem in economics [1] and control theory [4]. From a probabilistic point of view random sets have a rather well developed theory [7].

M is a set, a fuzzy subset of M is a function $u:M\to[0,1]$. The set of all fuzzy subsets of M, F(M) is a completely distribution lattice which includes the ordinary subsets of M. For any fuzzy subset $u:M\to[0,1]$ denote by $L_\alpha(u)=\{m\in M;u(m)\geq\alpha\}$ $\alpha\in[0,1]$ is the α-level set of u.

If M is a vector space a fuzzy subset $u\in F(M)$ is called a fuzzy Convex subset if
$u(\lambda m_1+(1-\lambda)m_2)\geq \min[u(m_1),u(m_2)]$ for every $m_1, m_2\in M$, $\lambda\in[0,1]$.

If X is a reflexive Banach space, in order to extend the Hausdorff distance we shall consider the subset $F_0(X)$ of F(X) containing all fuzzy sets $u:X\to[0,1]$ with properties

i) u is upper semi continuous.
ii) u is fuzzy convex.
iii) $L_\alpha(u)$ is compact for every $\alpha\neq0$.

If $u,v\in F_0(X)$ define the distance between u and v by
$d(u,v)=\sup_{\alpha>0} d_H(L_\alpha(u),L_\alpha(v))$Where d_H denotes the Hausdorff distance.

Let X be a normed space, and u be an open subset of X. Let y be a reflexive Banach space. By a fuzzy function we mean a function $F:u\to F_0(y)$ such a function associates to each point $x\in U$ a fuzzy subset F(x) of y clearly such fuzzy functions generalizes set valued function $u\to Q(y)$.

III. LIMIT THEOREMS

LEMMA: 3.1

If the fuzzy states Fs is recurrent and Fs→Fr, then Fr is recurrent and $f_{FsFr}=f_{FrFs}=1$.

Proof:

Assume Fs≠Fr for otherwise there is nothing to prove.

Since $f_{FsFr}>0$ there exists no such that $P_{FsFr}{}^{(n_0)}>0$ and

$P_{FsFr}{}^{(m)}=0$ for $0<m<n_0$. (3.1)

Since $P_{FsFr}{}^{(n_0)}>0$ we can find states $F_{1i}F_{i2}\ldots\ldots F_{in0-1}$ such that

$P_{FsFi1} \dots \dots . P_{FsFin_0 -1} > 0$ and none of the states $F_{i1} F_{i2} \dots \dots F_{in_0 -1}$ equal Fs or Fr, for if one of them did equal Fs or Fr it would be possible to go from Fs to Fr with positive probability in fewer then n_0 steps in contradiction to (3.1)

Suppose f_{FrFs} <1. Then a Markov chain staring from I has positive probability 1-f_{FrFs} of never hitting Fs and that implies it has positive probability$P_{FrF s_1} \dots \dots . P_{Fr n_0 -1 Fr} (1 - f_{FrFs})$ of visiting the states $F_{r1} F_{r2} \dots \dots F_{rn_0 -1}$, Fr successively in the first n_0 steps and never return to Fs after n_0 steps. But if this happens then the fuzzy Markov chain never return to Fs at any time n>1 and that contradict the fact that Fs is recurrent. So f_{FrFs}=1. Since f_{FrFs}=1 there exists n_1 such that$P_{FrFs}{}^{(n_1)} > 0$.

Now

$$P_{FrFr}{}^{(n_1+n+n_2)} \geq P_{FrFs}{}^{(n_1)} P_{FsFs}{}^{(n)} P_{FsFr}{}^{(n_0)}$$

and

hence

$$\sum_{n=1}^{\infty} P_{FrFr}{}^{(n)} \geq \sum_{n=1}^{\infty} P_{FrFr}{}^{(n_1+n+n_2)}$$

$$\geq P_{FrFs}{}^{(n_1)} P_{FsFr}{}^{(n_0)} \sum_{n=1}^{\infty} P_{FsFs}{}^{(n)} = \infty$$

Hence Fr is recurrent.

Since Fr is recurrent and Fr→Fs (f_{FrFs} =1) from the first part of the proof it follows that f_{FsFr}=1.

THEOREM: 3.1

$$P_{FrFs}{}^{(n)} = \sum_{m=1}^{\infty} f_{FrFs}{}^{(m)} P_{FsFs}{}^{(n-m)} \, for\ all\ m = 1,2, \dots . n$$

Proof:

$$P_{FrFs}{}^{(n)} = \bigcup_{\alpha \in (0,1]} \alpha (P_{FrFs}{}^{(n)})_\alpha$$

$$= \bigcup_{\alpha \in (0,1]} \alpha (P[X_n = F_s | X_0 = F_r])_\alpha$$

$$= \bigcup_{\alpha \in (0,1]} \alpha P[X_n = (F_s)_\alpha | X_0 = (F_r)_\alpha]$$

$$= \sum_{m=1}^{\infty} \bigcup_{\alpha \in (0,1]} \alpha P[X_n = (F_s)_\alpha X_m = (F_s)_\alpha X_{m-1} \neq (F_s)_\alpha \dots \dots X_1 \neq (F_s)_\alpha | X_0 = (F_r)_\alpha]$$

We take $X_m = (F_s)_\alpha = A$

$$X_m = (F_s)_\alpha X_{m-1} \neq (F_s)_\alpha \dots \dots X_1 \neq (F_s)_\alpha = B_m \ and\ X_0 = (F_r)_\alpha = c$$

$$P_{FrFs}{}^{(n)} = \sum_{m=1}^{n} P[AB_m | c]$$

Where B_m are disjoint and $\bigcup_{m-1}^{n} B_m \supset A$

Hence

$$P_{FrFs}{}^{(n)} = \sum_{m=1}^{n} \frac{P[AB_m] P[B_m c]}{P[c] P[AB_m c]}$$

$$= \sum_{m=1}^{n} P[A | B_m c] P[B_m | c]$$

$$= \sum_{m=1}^{\infty} \bigcup_{\alpha \in (0,1]} \alpha P[X_n = (F_s)_\alpha | X_m = (F_s)_\alpha X_{m-1} \neq (F_s)_\alpha \dots \dots X_1 \neq (F_s)_\alpha, X_0 = (F_r)_\alpha]$$

$$\bigcup_{\alpha \in (0,1]} \alpha P[X_n = (F_s)_\alpha X_m = (F_s)_\alpha X_{m-1} \neq (F_s)_\alpha \dots \dots X_1 \neq (F_s)_\alpha | X_0 = (F_r)_\alpha]$$

$$= \sum_{m=1}^{\infty} \bigcup_{\alpha \in (0,1]} \alpha P[X_n = (F_s)_\alpha | X_m = (F_s)_\alpha] \, f_{FrFs}{}^{(m)}$$

$$= \sum_{m=1}^{\infty} \bigcup_{\alpha\in(0,1]} \alpha\, P_{FsFr}{}^{(n-m)} f_{FrFs}{}^{(m)}$$

$$= \sum_{m=1}^{\infty} P_{FsFr}{}^{(n-m)} f_{FrFs}{}^{(m)}$$

THEOREM: 3.2 (LIMIT THEOREM)

Let Fs be a fixed state in a fuzzy Markov chain and Fr be an arbitrary state.

Then as n→∞.

(i) If Fs is transient then $P_{FsFr}{}^{(n)}$→0 as n→∞.

(ii) If Fs is null recurrent then $P_{FsFr}{}^{(n)}$→0

(iii) If Fs is positive recurrent and the Markov chain is aperiodic then $P_{FsFr}{}^{(n)} \to \frac{f_{FsFr}}{\mu_{Fs}}$

Proof:

By theorem 3.1

$$P_{FrFs}{}^{(n)} = \bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(n)})_\alpha$$

$$= \sum_{m=1}^{n} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(m)})_\alpha (P_{FsFs}{}^{(n-m)})_\alpha$$

$$= \sum_{m=1}^{n'} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(m)})_\alpha (P_{FsFs}{}^{(n-m)})_\alpha + \sum_{m=n'+1}^{n} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(m)})_\alpha (P_{FsFs}{}^{(n-m)})_\alpha \qquad (3.2)$$

Where n<n'<n; (n≥1)

For $\in > 0$ take n' and n so large that

$$\sum_{m=n'+1}^{n} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(m)})_\alpha < \in \qquad (3.3)$$

When Fs is transient or null recurrent take n so large that

$$\bigcup_{\alpha\in(0,1]} \alpha\, (P_{FrFs}{}^{(n-m)})_\alpha < \in for\ all\ 0 \le m < n' < n$$

By (3.2) and(3.3) we have

$$0 \le \bigcup_{\alpha\in(0,1]} \alpha\, (P_{FrFs}{}^{(n-m)})_\alpha - \sum_{m=n'+1}^{n} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(m)})_\alpha (P_{FsFs}{}^{(n-m)})_\alpha$$

$$= \sum_{m=n'+1}^{n} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(m)})_\alpha (P_{FsFs}{}^{(n-m)})_\alpha$$

$$\le \sum_{m=n'+1}^{n} \alpha (f_{FrFs}{}^{(m)})_\alpha < \in \qquad (3.4)$$

$$0 \le \lim_{n\to\infty} \bigcup_{\alpha\in(0,1]} \alpha\, (P_{FrFs}{}^{(n)})_\alpha$$

$$\le \epsilon + \epsilon \sum_{m=n'+1}^{n} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(m)})_\alpha \qquad from \quad (3.4)$$

$$\le \epsilon + \epsilon$$

$$= 2\epsilon\ for\ all\ \epsilon > 0$$

Therefore $\bigcup_{\alpha\in(0,1]} \alpha\, (P_{FrFs}{}^{(n)})_\alpha \to 0\ as\ n \to \infty$

(iii)Give that, the fuzzy state Fs is positive recurrent and the fuzzy Markov chain is aperiodic.

Take $n \to \infty$ and n' fixed.

Then

$$0 \le \lim_{n\to\infty} \bigcup_{\alpha\in(0,1]} \alpha\, (P_{FrFs}{}^{(n)})_\alpha - \lim_{n\to\infty} \sum_{m=1}^{n'} \bigcup_{\alpha\in(0,1]} \alpha\, (f_{FrFs}{}^{(n)})_\alpha (P_{FsFs}{}^{(n-m)})_\alpha$$

$< \in By$ (3.4)

$$= \lim_{n\to\infty} \bigcup_{\alpha\in(0,1]} \alpha\,({P_{FrFs}}^{(n)})_\alpha - \sum_{m=1}^{n'} \bigcup_{\alpha\in(0,1]} \alpha\,({f_{FrFs}}^{(n)})_\alpha \frac{1}{\mu_{Fs}}$$

$$= \lim_{n\to\infty} \bigcup_{\alpha\in(0,1]} \alpha\,({P_{FrFs}}^{(n)})_\alpha - \frac{1}{\mu_{Fs}} \sum_{m=1}^{n'} \bigcup_{\alpha\in(0,1]} \alpha\,({f_{FrFs}}^{(n)})_\alpha$$

$<\in$

Take $n'\to\infty$

$$0 \le \lim_{n\to\infty} \bigcup_{\alpha\in(0,1]} \alpha\,({P_{FrFs}}^{(n)})_\alpha - \frac{1}{\mu_{Fs}} \sum_{m=1}^{n'} \bigcup_{\alpha\in(0,1]} \alpha\,(f_{FrFs})_\alpha$$

$$\bigcup_{\alpha\in(0,1]} \alpha\,({P_{FrFs}}^{(n)})_\alpha \to \frac{\bigcup_{\alpha\in(0,1]} \alpha\,(f_{FrFs})_\alpha}{\mu_{Fs}}$$

$$(ie) \quad ({P_{FrFs}}^{(n)})_\alpha \to \frac{(f_{FrFs})_\alpha}{\mu_{Fs}}$$

IV. STATIONARY DISTRIBUTION

DEFINITION: 4.1

A probability distribution is $\{V_{Fs}\}$ with $V_{Fs} \ge 0$ $\sum_{Fs} V_{Fs} = 1$ is called a stationary distribution for a Markov chain with transition matrix P_{FrFs} if

$$V_{Fs} = \sum_{Fr} V_{Fr}\, P_{FrFs}$$

$$= \sum_{Fr} V_{Fr} \bigcup_{\alpha\in(0,1]} \alpha\,(P_{FrFs})_\alpha$$

$$= \sum_{Fr} \sum_{Fk} V_{Fk} \bigcup_{\alpha\in(0,1]} \alpha\,(P_{FkFr})_\alpha (P_{FrFs})_\alpha$$

$$= \sum_{Fk} V_{Fk} \sum_{Fr} \bigcup_{\alpha\in(0,1]} \alpha\,(P_{FkFr})_\alpha (P_{FrFs})_\alpha$$

$$= \sum_{Fk} V_{Fk} \bigcup_{\alpha\in(0,1]} \alpha\,({P_{FkFs}}^{(2)})_\alpha$$

....

$$= \sum_{Fk} V_{Fk} \bigcup_{\alpha\in(0,1]} \alpha\,({P_{FkFs}}^{(n)})_\alpha$$

$$= \sum_{Fk} V_{Fk}\, {P_{FkFs}}^{(n)}$$

Suppose a stationary distribution

$\pi = (\pi_1, \pi_2, \dots\dots)$ exists. Also suppose

$$\lim_{n\to\infty} \bigcup_{\alpha\in(0,1]} \alpha\,({P_{FrFs}}^{(n)})_\alpha = \pi_{Fs} \ge 0 \; for\ all\ Fr \ge 1.$$

Then π is called the steady state distribution of the Markov chain with Transition matrix (P_{FrFs}).

THEOREM 4.1

Let a Fuzzy Markov chain is irreducible, aperiodic and positive. Then

(i) $\lim_{n\to\infty} {P_{FrFs}}^{(n)} = \pi_{Fs}$

(ii) $\pi_{Fs} > 0$ $\sum_{Fs} \pi_{Fs} = 1$

(iii) $\pi_{Fs} = \sum_{Fs\in s} \pi_{Fk} P_{FkFs}$

More over (ii) and (iii) determine $\{\pi_{Fs}\,, Fs \in s\}\ Completely.$

Proof:

(i) The Proof of (i) follows from theorem 2.2 and the lemma.

(ii) $\pi_{Fs} = \frac{1}{\mu_{Fs}} > 0$

Suppose S_M is a subset of the state space S with exactly M states.

Now,

$$\sum_{Fs\in S_M} {P_{FrFs}}^{(n)}$$

$$= \sum_{Fs \in S_M} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FrFs}{}^{(n)})_\alpha$$
$$\leq \sum_{Fs \in S} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FrFs}{}^{(n)})_\alpha$$
$$= 1$$

Let n→∞ then

$$\sum_{Fs \in S_M} \pi_{Fs} \leq 1$$

Then taking limit M→∞

$$\sum_{Fs \in S_M} \pi_{Fs} \leq 1 \qquad (4.1)$$

Then taking limit

$$\sum_{Fs \in S_M} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FrFk}{}^{(n)})_\alpha\, (P_{FkFs})_\alpha$$
$$\leq \sum_{Fs \in S_M} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FrFk}{}^{(n)})_\alpha\, P_{FkFs}$$
$$= \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FrFs}{}^{(n+1)})_\alpha$$

Let n→∞ then

$$\lim_{n \to \infty} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FrFs}{}^{(n+1)})_\alpha$$
$$= \sum_{Fs \in S_M} \pi_{Fk} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FkFs})_\alpha$$
$$\leq \pi_{Fs}$$

Then letting M→∞ we get

$$\sum_{Fs \in S_M} \pi_{Fk} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FkFs})_\alpha$$
$$\leq \pi_{Fs} \qquad (4.2)$$

$$\sum_{Fs \in S} \pi_{Fs} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FsFr}{}^{(2)})_\alpha$$
$$= \sum_{Fs \in S} \pi_{Fs} \bigcup_{\alpha \in (0,1]} \alpha (P_{FsFk})_\alpha (P_{FkFr})_\alpha$$
$$= \bigcup_{\alpha \in (0,1]} \sum_{Fs \in S} \pi_{Fs} \alpha (P_{FsFk})_\alpha (P_{FkFr})_\alpha$$
$$= \bigcup_{\alpha \in (0,1]} \sum_{Fs \in S} \left(\sum_{Fs \in S} \pi_{Fs} \alpha (P_{FsFk})_\alpha \right) (P_{FkFr})_\alpha$$
$$\leq \bigcup_{\alpha \in (0,1]} \sum_{Fs \in S} \pi_k \alpha (P_{FkFr})_\alpha$$
$$= \sum_{Fs \in S} \pi_{Fk} P_{FkFr}$$
$$\leq \pi_{Fr}$$

By induction

$$\sum_{Fs \in S} \pi_{Fs} \bigcup_{\alpha \in (0,1]} \alpha\, (P_{FsFr}{}^{(n)})_\alpha \leq \pi_{Fr}\ for\ all\ n \geq 1; Fs \in S$$

Now

$$\pi_{Fk} = \pi_{Fk} \left(\sum_{j \in S} P_{FkFs}{}^{(n)} \right)$$
$$\left(\sum_{j \in S} P_{FkFs}{}^{(n)} = 1 \right)$$

$$\sum_{Fk\in S}\pi_{Fk} = \sum_{Fk\in S}\sum_{Fs\in S}\bigcup_{\alpha\in(0,1]}\alpha\pi_k\,({P_{FkFs}}^{(n)})_\alpha$$
$$= \sum_{Fk\in S}\sum_{Fs\in S}\pi_k\bigcup_{\alpha\in(0,1]}\alpha\,({P_{FkFs}}^{(n)})_\alpha$$

By Fubinis theorem.

Suppose
$$\sum_{Fk\in S}\pi_{Fk}\bigcup_{\alpha\in(0,1]}\alpha\,{P_{FkFs}}^{(n)} < \pi_{Fs}$$
Then
$$\sum_{Fk\in S}\sum_{Fs\in S}\pi_k\bigcup_{\alpha\in(0,1]}\alpha\,({P_{FkFs}}^{(n)})_\alpha < \sum_{Fs\in S}\pi_{Fs}\ and$$
$$\sum_{Fk\in S}\pi_{Fk} < \sum_{Fs\in S}\pi_{Fs}$$

Which is a Contradiction.

Thus
$$\sum_{Fk\in S}\pi_{Fk}\bigcup_{\alpha\in(0,1]}\alpha\,({P_{FkFs}}^{(n)})_\alpha = \sum_{Fk\in S}\pi_{Fk}{P_{FkFs}}^{(n)}$$
$$= \pi_{Fs}\ for\ n\geq 1 \qquad (4.3)$$

In particular for $n\geq 1$ $\sum_{Fs\in S}\pi_{Fs}P_{FsFr} = \pi_{Fr}$

This Proves (iii).

Moreover by Lebesgue Dominated convergence theorem and part(i) letting n→∞ in (4.3)
$$\sum_{Fs\in S}\pi_{Fs}\pi_{Fr} = \pi_{Fr}$$
Now $\pi_{Fr} > 0$ that gives
$$\sum_{Fs\in S}\pi_{Fs} = 1$$
To show that the solution given by (ii) and (iii) is unique. Suppose that $\{x_{Fr}, Fr\in S\}$ is another such solution satisfying $x_{Fr}>0$
$$\sum_{Fs\in S}\pi_{Fs} = 1$$
and
$$x_{Fr} = \sum_{Fs\in S}x_{Fs}P_{FsFr}$$
$$= \sum_{Fs\in S}x_{Fs}\bigcup_{\alpha\in(0,1]}\alpha\,(P_{FsFr})_\alpha$$
$$= \sum_{Fs\in S}\left(\sum_{Fk\in S}\bigcup_{\alpha\in(0,1]}\alpha\,x_{Fk}\,(P_{FkFs})_\alpha\right)(P_{FsFr})_\alpha$$
$$= \sum_{Fs\in S}x_{Fk}\left(\sum_{Fk\in S}\bigcup_{\alpha\in(0,1]}\alpha\,(P_{FkFs})_\alpha\,(P_{FsFr})_\alpha\right)$$

(By Fubinis theorem)

$$= \sum_{Fs\in S}x_{Fk}{P_{FkFr}}^{(2)}$$
=………………
$$= \sum_{Fs\in S}x_{Fk}{P_{FkFr}}^{(n)}$$
By the Lebesgue Dominated Convergence theorem, Letting n→∞
$$x_{Fr} = \sum_{Fk\in S}x_{Fs}\,\pi_{Fr} = \pi_{Fr}\sum_{Fk\in S}x_{Fs} = \pi_{Fr}\ for\ all\ Fr\in S$$
Thus the solution $\{\pi_i \quad i\in S\}$ is unique.

THEOREM: 4.2
A Fuzzy Markov chain remains Markov if time is reversed.
$P(X_n=F_{rn}|X_{n+1}=F_{rn+1} \ldots\ldots X_{n+k}=F_{rn+k})$
$= P(X_n=F_{rn}|X_{n+1}=F_{rn+1})$
Proof:

$$P(X_{n-1} = Fr_{n-1}|X_n = Fr_n, X_{n+1} = Fr_{n+1} \ldots\ldots)$$
$$= \bigcup_{\alpha\in(0,1]} \alpha\, P(X_{n-1} = (Fr_{n-1})_\alpha|X_n = (Fr_n)_\alpha, X_{n+1} = (Fr_{n+1})_\alpha \ldots\ldots)$$
$$= \frac{\bigcup_{\alpha\in(0,1]} \alpha P(X_{n-1} = Fr_{n-1}, X_n = Fr_n, X_{n+1} = Fr_{n+1} \ldots\ldots)}{\bigcup_{\alpha\in(0,1]} \alpha\, P(X_n = Fr_n, X_{n+1} = Fr_{n+1} \ldots\ldots)}$$
$$= \frac{\bigcup_{\alpha\in(0,1]} \alpha P(X_{n+1} = (Fr_{n+1})_\alpha, X_{n+2} = (Fr_{n+2})_\alpha \ldots|X_n = (Fr_n)_\alpha, X_{n-1} = (Fr_{n-1})_\alpha) \,.\, \bigcup_{\alpha\in(0,1]} \alpha\, P(X_n = (Fr_n)_\alpha, X_{n-1} = (Fr_{n-1})_\alpha)}{\bigcup_{\alpha\in(0,1]} \alpha\, P(X_{n+1} = (Fr_{n+1})_\alpha, X_{n+2} = (Fr_{n+2})_\alpha \ldots|X_n = (Fr_n)_\alpha) \,.\, \bigcup_{\alpha\in(0,1]} \alpha\, P(X_n = (Fr_n)_\alpha)}$$
$$= \bigcup_{\alpha\in(0,1]} \alpha\, P(X_{n-1} = (Fr_{n-1})_\alpha|X_n = (Fr_n)_\alpha)$$
$$= P(X_{n-1} = (Fr_{n-1})|X_n = (Fr_n))$$

THEOREM: 4.3
In a Fuzzy Markov chain if the present is specified then the past is independent of the future in the following sense.

$$P(X_n = Fr_n\; X_{Fk} = Fr_k|X_m = Fr_m) = P(X_n = (Fr)_n|X_m = (Fr)_m)P(X_k = (Fr)_k|X_m = (Fr)_m)$$

Proof:
By the Chain rule of conditional probabilities

$$P(X_n = Fr_n\; X_{Fk} = Fr_k|X_m = Fr_m)$$
$$= \frac{\bigcup_{\alpha\in(0,1]} \alpha P(X_n = (Fr_n)_\alpha\; X_k = ((Fr)_k)_\alpha X_m = ((Fr)_m)_\alpha)}{\bigcup_{\alpha\in(0,1]} \alpha\, P(X_m = ((Fr)_m)_\alpha)}$$
$$= \frac{\bigcup_{\alpha\in(0,1]} \alpha P(X_n = (Fr_n)_\alpha\; |X_k = ((Fr)_k)_\alpha X_m = ((Fr)_m)_\alpha)}{\bigcup_{\alpha\in(0,1]} \alpha\, P(X_m = (Fr_m)_\alpha)}$$
$$= \frac{\bigcup_{\alpha\in(0,1]} \dfrac{\alpha P(X_n = (Fr_n)_\alpha|X_m = (Fr_m)_\alpha)P(X_m = (Fr_m)_\alpha|\, X_k = (Fr_k)_\alpha)}{P(\,X_k = (Fr_k)_\alpha)}}{\bigcup_{\alpha\in(0,1]} \alpha\, P(X_m = (Fr_m)_\alpha)}$$

By Markov Property

$$= \frac{\bigcup_{\alpha\in(0,1]} \alpha P(X_n = (Fr_n)_\alpha|X_m = (Fr_m)_\alpha)P(X_m = (Fr_m)_\alpha|\, X_k = (Fr_k)_\alpha)}{\bigcup_{\alpha\in(0,1]} \alpha\, P(X_m = (Fr_m)_\alpha)}$$
$$= \bigcup_{\alpha\in(0,1]} \alpha P(X_n = (Fr_n)_\alpha|X_m = (Fr_m)_\alpha)P(\,X_k = (Fr_k)_\alpha|X_m = (Fr_m)_\alpha)$$

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